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ON THE VOLUME OF MANIFOLDS ALL OF WHOSE GEODESICS ARE CLOSED

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1. C_L-manifolds

A riemannian manifold (M, g) will be called a C_L-manifold if all the geodesics on M are closed and have length $2\pi L$, i.e., if all the orbits of the geodesic flow on the unit tangent bundle $U(M, g)$ are periodic with least period $2\pi L$. It is a problem of some interest to characterize these manifolds, which are the "simple harmonic oscillators" of riemannian geometry.

The best known examples of C_L-manifolds are the symmetric spaces of rank one, or SC-manifolds, as they are called by Berger [1, III. 4]. These are the spheres (S^n, can), projective spaces $(P^n(K), \text{can})$ for $k = \boldsymbol{R}, \boldsymbol{C}$, or $\boldsymbol{H}$, and the Cayley projective plane $(P^2(\Gamma), \text{can})$, with their canonical metrics. The spheres are C_1-manifolds, and the projective spaces are, with Berger's normalization, $C_{1/2}$-manifolds.

Zoll (see [1, IV. 8]) in 1903 constructed examples of non-standard C_L-metrics on S^2 (surfaces of revolution), and Blaschke [3, p. 233] gives an example, due to Thomsen, of a C_L-metric on S^2 with no nontrivial isometries. These constructions can be carried out on higher dimensional spheres as well. If one strengthens the C_L condition to require that the geodesics be simple closed curves on M, then a theorem of Green (see [1, VIII. 9]) states that any such simple C_L-metric on $P^2(\boldsymbol{R})$ has constant curvature. Furthermore, it is a theorem of Bott (see [1, IV. 6]) that every simple C_L-manifold has the same integer cohomology ring as some SC-manifold. In fact, this result requires only that all the geodesics through a single point of M be simply closed with the same length. The earliest topological study of C_L-manifolds seems to be that of Reeb [7], who proved, among other things, that the product of two spheres of different odd dimensions cannot carry a C_L-metric.

The aim of the present paper is to demonstrate the following geometric result.

Theorem A. *If (M, g) is an n-dimensional C_L-manifold, then the ratio*

$$i(M, g) = \frac{\text{vol}\,(M, g)}{L^n \,\text{vol}\,(S^n, \text{can})}$$

is an integer.

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We will actually prove the following theorem, of which Theorem A is an immediate consequence.

Theorem B. *If (M, g) is an n-dimensional C_L-manifold, then the real number $j(M, g)$ defined by the equation*

$$\text{vol}\,(M, g) = \frac{(2\pi L)^n \cdot j(M, g)}{(n-1)!\,\text{vol}\,(S^{n-1}, \text{can})} \tag{1}$$

is an even integer.

To prove Theorem A from Theorem B, one has merely to check, using the values of $\text{vol}\,(S^{n-1}, \text{can})$ [1, VI. 7], that $j(S^n, \text{can}) = 2$; then set $i(M, g) := \frac{1}{2}j(M, g)$.

Remarks

1. The proof of Theorem B, contained in the following two sections of this paper, identifies the integer $j(M, g)$ as a topological invariant of the fibration of $U(M, g)$ by the orbits of the geodesic flow.

2. Using Gysin sequences one can prove that $j(M, g) = 2$ and $i(M, g) = 1$ if M is an even-dimensional sphere. It would be interesting to prove that $i(M, g)$ is independent of g when M is any SC-manifold. This may be a step in the direction of generalizing the theorem of Green mentioned above.

3. In the succeeding paper in this journal [2], Marcel Berger proves the following application of Theorem A. Let g be a Kählerian metric on $P^n(C)$, compatible with the standard complex structure. Suppose that the distance to the first conjugate point in each direction from each point on $P^n(C)$ is $\frac{1}{2}\pi$. Then, at least if g is sufficiently near the canonical metric in the C^0 topology, $(P^n(C), g)$ is isometric to $(P^n(C), \text{can})$.

4. Funk [4, p. 283] remarks that the area of a C_1-surface of revolution must be 4π. Otherwise, our result seems to be new even for $M = S^2$.

5. An amusing consequence of Theorem A is that one cannot apply a slight perturbation to (S^n, can) to make the geodesics close only after $k > 1$ "revolutions", for then the volume of the manifold would have to be multiplied by k^n.

6. For reference, we present the following formulas, obtained from the calculations in [1, VI. 7]:

$$i(S^n, \text{can}) = 1\,, \qquad i(P^n\boldsymbol{R}, \text{can}) = 2^{n-1}\,,$$

$$i(P^n\boldsymbol{C}, \text{can}) = \binom{2n-1}{n-1}, \qquad i(P^n\boldsymbol{H}, \text{can}) = \frac{1}{2n+1}\binom{4n-1}{2n-1},$$

$$i(P^2\Gamma, \text{can}) = 39\,.$$

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2. Proof of Theorem B

The unit tangent bundle $U(M, g)$ of a riemannian manifold carries the following geometric objects:

the geodesic spray G, [1, IV. 2],
the canonical one-form α, [1, III. 6],
the canonical two-form $d\alpha$, [1, III. 6],
the riemannian metric $\bar{\bar{g}}$, [1,V. 2.4],
the volume element $\bar{\bar{\theta}}$, [1, V. 2.4].

These objects satisfy the following relations:

i. $\left|\dfrac{\alpha \wedge (d\alpha)^{n-1}}{(n-1)!}\right| = \bar{\bar{\theta}}$, [1, V. 2.5],

ii. $\operatorname{vol}(U(M, g), \bar{\bar{g}}) = \operatorname{vol}(M, g) \cdot \operatorname{vol}(S^{n-1}, \text{can})$, [1, V. 2.13],

iii. the flow of G leaves α invariant [1, IV. 3.10],

iv. $\alpha(G) \equiv 1$, [1, p. 125],

v. the null space (characteristic distribution) of $d\alpha$ is generated by G, [5, Thm. 5.9].

Since the orbits of G are all periodic with period $2\pi L$, the vector field $2\pi LG$ generates a free action of $S^1 = \boldsymbol{R}/\boldsymbol{Z}$ on $U(M, g)$, with quotient a manifold $C(M, g)$. The projection $U(M, g) \xrightarrow{p} C(M, g)$ is a principal bundle with structure group S^1. Relations iii and iv above mean that $\alpha/(2\pi L)$ is a connection form on this bundle, and $d\alpha/(2\pi L)$ is the curvature form. There is then a uniquely determined form Ω on $C(M, g)$ such that $p^*\Omega = d\alpha/(2\pi L)$; the de Rham cohomology class $[\Omega] \in H^2(C(M, g); \boldsymbol{R})$ is the image under the coefficient homomorphism $\rho_2\colon H^2(C(M, g); \boldsymbol{Z}) \to H^2(C(M, g); \boldsymbol{R})$ of the Euler class $e(p)$ of the bundle p. (We identify the group S^1 with $SO(2)$.) Then $[\Omega^{n-1}] = [\Omega]^{n-1}$ is the image of $[e(p)]^{n-1}$ under the coefficient homomorphism ρ_{2n-2}.

By relation v, the form Ω is nonsingular on $C(M, g)$, which is oriented by Ω^{n-1}. Denoting by $[C(M, g)]$ the fundamental $(2n-2)$-cycle, we have

$$\int_{C(M,g)} \Omega^{n-1} = \langle [e(p)]^{n-1}, [C(M, g)] \rangle \ . \tag{2}$$

Let $j(M, g)$ be the quantity on either side of (2). The left hand side of the equation is positive, and the right hand side is an integer, so $j(M, g)$ is a positive integer.

The argument up to here is essentially contained in [7]. At this point, we use the Fubini theorem for fibrations [1, 0.3.17] to calculate

$$\begin{aligned}\operatorname{vol}(U(M,g);\bar{\bar{g}}) &= \int_{U(M,g)} \frac{\alpha\wedge(d\alpha)^{n-1}}{(n-1)!} \\ &= \frac{1}{(n-1)!}\int_{U(M,g)} \alpha\wedge p^*(2\pi L\Omega)^{n-1} \\ &= \frac{(2\pi L)^{n-1}}{(n-1)!}\int_{x\in C(M,g)} \left[\int_{p^{-1}(x)}\alpha\right]\Omega^{n-1}.\end{aligned}$$

By relation iv, $\int_{p^{-1}(x)}\alpha = 2\pi L$ for each x, so the above expression becomes

$$\frac{(2\pi L)^n}{(n-1)!}\int_{C(M,g)}\Omega^{n-1} = \frac{2\pi L}{(n-1)!}j(M,g)\ .$$

Combining this with relation ii shows that $j(M,g)$ satisfies (1). To complete the proof of Theorem B, it remains only to show that $j(M,g)$ is even. This is done in the next section.

3. Involutions and evenness

Let $\xi\colon P\to B$ be a principal bundle with structure group $SO(2)$. By means of the embedding $SO(2)\to O(2)$, we can consider ξ as a bundle with fibre $SO(2)$ and structure group $O(2)$. Let $\beta\colon \xi\to\xi$ be a mapping of $O(2)$ bundles (see [8, 2.5] for a definition) with $\beta^2=$ identity and such that the induced mapping $\gamma\colon B\to B$ has no fixed points. Suppose further that B is an orientable manifold of dimension $2n$.

Proposition. *The class $[e(\xi)]^n$ is an even multiple of the generator of $H^{2n}(B;Z)\approx Z$.*

Proof. By [6, 4.11.2 III], $[e(\xi)]^n = e(n\xi)$ where $n\xi$ is the $SO(2n)$ bundle obtained by taking the n-fold Whitney sum of ξ with itself. The involution β induces an involution $n\beta\colon n\xi\to n\xi$ of $O(2n)$ bundles; therefore there is an $O(2n)$ bundle $\overline{n\xi}$ over the quotient manifold $\bar{B}=B/\gamma$ such that $n\xi=\pi^*\overline{n\xi}$, $\pi\colon B\to\bar{B}$ being the projection. The Whitney classes of $n\xi$ and $\overline{n\xi}$ satisfy the relation $w_{2n}(n\xi)=\pi^*w_{2n}(\overline{n\xi})$ [6, p. 73]. Since π is a double covering, it induces the zero map from $H^{2n}(\bar{B};Z_2)$ to $H^{2n}(B;Z_2)$, so $w_{2n}(n\xi)=0$. But $w_{2n}(n\xi)$ is the mod 2 reduction [6, p. 73] of $e(n\xi)=e(\xi)]^n$, so $[e(\xi)]^n$ is even. q.e.d.

To apply this Proposition to Theorem 2, we use the involution $h_{-1}\colon U(M,g)\to U(M,g)$ defined by multiplying each tangent vector by -1. Since G is a spray, G is h_{-1}-related to $-G$, so h_{-1} is an $O(2)$ bundle mapping. Finally, the induced map on $C(M,g)$ has no fixed points, because a geodesic cannot double back upon itself in the reverse direction.

Hence the class $[e(p)]^{n-1}$ is an even multiple of the generator of $H^{2n-2}(C(M,g);Z)$, and $j(M,g)=\langle[e(p)]^{n-1},[C(M,g)]\rangle$ is an even integer.

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